PLANE CUBICS WITH A GIVEN QUADRANGLE OF INFLEXIONS.

By B. M. Turner.

That every non-singular cubic has nine points of inflexion, lying in related positions on the curve, is a classical fact in mathematics. Of these points four may be chosen arbitrarily; and when such a quadrangle is fixed, the finding of the positions of the remaining five presents a question worthy of consideration. It appears that all the sets of five combine into a group of fifteen points whose relative positions with respect to the given four depend upon equianharmonic properties; but that the equianharmonic relations follow as a consequence of a combination of harmonic relations and hence, in a number of cases, the points may be determined by linear and quadratic constructions.*

It is also well known that four points of inflexion, no three collinear, impose eight conditions on a cubic and determine it as one of a singly infinite system; but, since only four of the conditions are linear while the other four are of the third degree, the system is not a pencil. It will be shown that the four points determine a system consisting of six pencils and that every two of the six have a fifth point of inflexion in common, that is, through every one of the fifteen points two of the pencils pass, and have consequently an inflexion.

I. Determination and Construction of the Remaining Five Inflexions.

Four points of inflexion of a cubic may be chosen arbitrarily and the conditions imposed by any one of the four are then independent of the conditions imposed by the other three. For a real cubic, however, the imaginary points of inflexion occur in conjugate pairs; hence for such a cubic, if no more than four of the inflexions are involved in the selection, the chosen quadrangle must consist of (1) two pairs of imaginary points, or (2) two real points and one imaginary pair. As a system consisting entirely of imaginary cubics is in itself of little interest, only these two quadrangles supplemented for symmetry by a third with four real vertices will be considered in this discussion. The results for the three cases can be stated in identical terms, as shown in the theorems given in § 3 (pp. 277–8).

* That the whole set of nine points depends only on quadratic constructions was virtually shown by Möbius (Gesammelte Werke, I, p. 437) in the determination of two quadrangles in- and circumscribed to one another. The eight vertices with the addition of the one common diagonal point form the desired set.

§ 1. GIVEN QUADRANGLE—TWO PAIRS OF IMAGINARY POINTS.

Let the two pairs of imaginary points to be taken as points of inflexion for a cubic be given as the intersections of two real lines with a conic (Fig. 1).

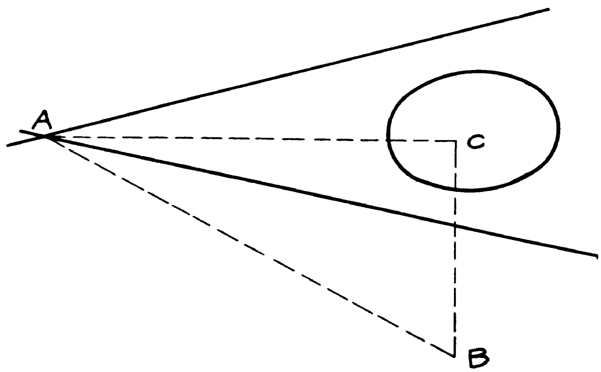

FIG. 1.

Then, as is known, the pencil of conics through the four points has a real common self-polar triangle with A, the intersection of the two given lines, as a vertex and BC, the polar line of A with respect to the given conic, as a side. It is further known that the remaining vertices B, C may be determined by a quadratic construction and hence, since every quadratic construction can be performed by means of the one given conic, the self-polar triangle may be constructed geometrically.*

As the relations to be noted are invariant under projection and any four points forming a quadrangle may be projected into the vertices of any other quadrangle, there is no loss of generality in choosing the points as $(i, \pm 1, \pm 1)$.† Then the self-polar triangle is the triangle of reference, $y \pm z = 0$ are the given lines, and the conic is one of the pencil

$$ax^2 + by^2 + cz^2 = 0$$

where $-a + b + c = 0$.

(i) *Determination of the Five Points.*

Since each component of the three pairs of sides of the quadrangle

$$y^2 - z^2 = 0, \qquad z^2 + x^2 = 0, \qquad x^2 + y^2 = 0,$$

passes through two of the given points, the six lines are inflexional axes‡

* Two pairs of imaginary points can of course be given by two conics; but if so the determination of the self-polar triangle, also of the real line pair, cannot be accomplished by quadratic constructions.

† Throughout this article the symbol i is used for $\sqrt{-1}$ taken positively. In the cases where an ambiguity enters, it is always preceded by the double sign.

‡ Inflexional axis: a line through three points of inflexion.

for every cubic inflected at the four points. For a real cubic it is known that the real sides of the quadrangle determined by two pairs of its imaginary points of inflexion are sides of the real inflexional triangle, the intersection of the lines forming one pair of imaginary sides of the quadrangle is a point of inflexion, and the intersection of the lines forming the other pair of imaginary sides is the point common to the three real harmonic polars. Hence a real cubic with inflexions at the given four points has the lines through A as sides of the real inflexional triangle, and a fifth inflexion lies either at B or C.

If the fifth point of inflexion is at C, the third side of the real inflexional triangle passes through this point and has an equation of the form $x + \alpha y = 0$, where α is a real number still to be determined. The point B is common to the three real harmonic polars; and the "line of reals,"* being the polar of B with respect to the triangle

$$y \pm z = 0, \qquad x + \alpha y = 0,$$

is $x + 3\alpha y = 0$. Hence since the inflexional axes concurrent with the real harmonic polars $(z \pm ix = 0)$ together with the line of reals form a second inflexional triangle, the desired cubic is represented by

$$(z^2 + x^2)(x + 3\alpha y) + \lambda(y^2 - z^2)(x + \alpha y) = 0;$$

and the four remaining points of inflexion, being given by the intersections of $x + 3\alpha y = 0$ with $y^2 - z^2 = 0$ and of $x + \alpha y = 0$ with $z^2 + x^2 = 0$, are

$$(3\alpha, \ -1, \ \pm 1), \qquad (\alpha, \ -1, \ \pm i\alpha).$$

These nine points, namely the four given points, the point C, and the four just found, are points of inflexion for a cubic if, and only if, they satisfy the conditions of collinearity represented by the rows, columns, three right- and three left-hand diagonals of the following scheme:†

$$\begin{array}{lll}
(3\alpha, \ -1, \ -1), & (\ i, \ \ 1, \ 1), & (i, \ -1, \ -1), \\
(\ i, \ \ 1, \ -1), & (3\alpha, \ -1, \ 1), & (i, \ -1, \ \ 1), \\
(\ \alpha, \ -1, \ -i\alpha), & (\ \alpha, \ -1, \ i\alpha), & (0, \ \ 0, \ \ 1).
\end{array}$$

This requires that $3\alpha^2 = 1$. Accordingly the four points are either

$$(\sqrt{3}, \ -1, \ \pm 1), \ (1, \ -\sqrt{3}, \ \pm i) \ \text{ or } \ (-\sqrt{3}, \ -1, \ \pm 1), \ (1, \ \sqrt{3}, \ \pm i);$$

and the cubic is a member of one of the two pencils:

$$(1) \qquad (z^2 + x^2)(x + \sqrt{3}y) + \lambda(y^2 - z^2)\left(x + \frac{1}{\sqrt{3}}y\right) = 0,$$

* The line through the three real points of inflexion.

† Hesse, *Crelle's Journal* (1849), Vol. 38, p. 257; also Clebsch, "Vorlesungen über Geometrie," p. 506.

(2) $$(z^2 + x^2)(x - \sqrt{3}y) + \lambda(y^2 - z^2)\left(x - \frac{1}{\sqrt{3}}y\right) = 0.$$

Similarly if a cubic have B as a fifth point of inflexion, the remaining four inflexions are either
$(\sqrt{3}, \pm 1, -1)$, $(1, \pm i, -\sqrt{3})$ or $(-\sqrt{3}, \pm 1, -1)$, $(1, \pm i, \sqrt{3})$; and the corresponding pencils have equations

(3) $$(y^2 - z^2)(z + \sqrt{3}x) + \lambda(x^2 + y^2)\left(z + \frac{1}{\sqrt{3}}x\right) = 0,$$

(4) $$(y^2 - z^2)(z - \sqrt{3}x) + \lambda(x^2 + y^2)\left(z - \frac{1}{\sqrt{3}}x\right) = 0.$$

For symmetry, A is considered as a fifth point of inflexion, and the set of nine points is completed by either
$(\pm 1, \sqrt{3}, i)$, $(\pm 1, i, -\sqrt{3})$ or $(\pm 1, -\sqrt{3}, i)$, $(\pm 1, i, \sqrt{3})$; but the pencils

(5) $$(x^2 + y^2)(y + i\sqrt{3}z) + \lambda(z^2 + x^2)\left(y + \frac{i}{\sqrt{3}}z\right) = 0,$$

(6) $$(x^2 + y^2)(y - i\sqrt{3}z) + \lambda(z^2 + x^2)\left(y - \frac{i}{\sqrt{3}}z\right) = 0,$$

are imaginary.

These results may be stated in the form of the theorem:

(1) If two pairs of imaginary points of inflexion of a plane cubic are fixed, a real fifth point of inflexion is fixed as one of three, and the complete group of nine is determined as one of six;

or in other words

(1′) Two pairs of imaginary points of inflexion determine a system of cubics consisting of six syzygetic pencils, four real and two imaginary; and every two of the six have a fifth point of inflexion in common.

(ii) *Relative Positions of the Inflexions.*

The curves of the system of cubics have in all nineteen points of inflexion: namely, the four common to all the curves and fifteen others of which every one is common to two of the six pencils. When the four common points are $(i, \pm 1, \pm 1)$, the fifteen are

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1),$$
$$(\sqrt{3}, \pm 1, \pm 1), \quad (\pm 1, \sqrt{3}, \pm i), \quad (\pm 1, \pm i, \sqrt{3}).$$

These values show that the points determine certain quadranges. The lines

$$y \pm z = 0, \quad z \pm ix = 0, \quad x \pm iy = 0$$

are the sides of the chosen quadrangle; while the sides of the determined quadrangles are

$$y \pm z = 0, \qquad z \pm \frac{1}{\sqrt{3}} x = 0, \qquad x \pm \sqrt{3} y = 0;$$

$$y \pm i\sqrt{3} z = 0, \qquad z \pm ix = 0, \qquad x \pm \frac{1}{\sqrt{3}} y = 0;$$

$$y \pm \frac{1}{\sqrt{3}} z = 0, \qquad z \pm \sqrt{3} x = 0, \qquad x \pm iy = 0;$$

Hence follows the theorem:

(2) The remaining points of inflexion of the cubics with two common pairs of imaginary inflexions form a group of fifteen, consisting of the three diagonal points of the common quadrangle and the vertices of three other quadrangles with the same diagonal points, each of the three new quadrangles having one pair of sides in common with the original quadrangle.

The sides of the quadrangles and their common diagonal triangle are further related. The six real lines through C considered as three pairs

$$x = 0,\ y = 0;\ x + \sqrt{3} y = 0,\ x - \frac{1}{\sqrt{3}} y = 0;\ x - \sqrt{3} y = 0,\ x + \frac{1}{\sqrt{3}} y = 0$$

form a pencil in elliptic involution with

$$x \pm iy = 0$$

as double lines. The lines of any one of the three pairs are harmonic with respect to the first lines of the other two pairs, and also with respect to the last lines. Furthermore either triad of lines

$$x = 0, \qquad x \pm \sqrt{3} y = 0 \qquad \text{or} \qquad y = 0, \qquad x \pm \frac{1}{\sqrt{3}} y = 0,$$

together with either double line, forms an equianharmonic system.* Thus the lines

$$x \pm \sqrt{3} y = 0, \qquad x \pm \frac{1}{\sqrt{3}} y = 0$$

satisfy a combination of harmonic and equianharmonic relations with respect to $x = 0$, $y = 0$, $x \pm iy = 0$. The equianharmonic properties, however, are simply a consequence of harmonic properties; for if we write

$$x = 0, \quad y = 0; \quad x + \beta y = 0, \quad x - \alpha y = 0; \quad x - \beta y = 0, \quad x + \alpha y = 0,$$

* Consider $x = 0$, $x \pm \sqrt{3} y = 0$, $x + iy = 0$. Let $X = x + \sqrt{3} y$, $Y = x - \sqrt{3} y$; then $x = 0$, $x + iy = 0$ are transformed respectively into $X + Y = 0$, $X - \omega Y = 0$; and the cross-ratio of the four lines $X = 0$, $Y = 0$, $X + Y = 0$, $X - \omega Y = 0$ is $-\omega^2$, where $\omega^3 = 1$. Similarly for the other combinations.

and impose the condition that $x = 0$, $x + \beta y = 0$ be harmonic with respect to $x - \beta y = 0$, $x + \alpha y = 0$, we find $\beta = 3\alpha$; and the condition that $x + 3\alpha y = 0$, $x - \alpha y = 0$ be harmonic with respect to $x \pm iy = 0$ shows that $3\alpha^2 = 1$.

Similar statements hold for the lines through B; and also for the set through A, except that in this case the involution is hyperbolic. Consequently the sides of the three quadrangles, and hence the vertices, are uniquely determined from the sides of the base quadrangle and its diagonal triangle by means of harmonic properties.

The equianharmonic properties furnish an analytic means of determining the points, and also serve to show the relative positions of the three quadrangles with respect to the four given points. The points equianharmonic to $(1, 0, 0)$ with respect to $(i, 1, 1)$, $(i, -1, 1)$ are $(\sqrt{3}, 1, 1)$ and $(3, -1, -1)$; and those equianharmonic with respect to $(i, -1, 1)$, $(i, 1, -1)$ are $(\sqrt{3}, -1, 1)$, $(\sqrt{3}, 1, -1)$. Thus the vertices of the real quadrangle $(\sqrt{3}, \pm 1, \pm 1)$ are the points on the lines $y \pm z = 0$ equianharmonic to $(1, 0, 0)$ with respect to $(i, \pm 1, \pm 1)$. Similarly $(\pm 1, \sqrt{3}, \pm i)$ are the points on $z \pm ix = 0$ and $(1, \pm i, \sqrt{3})$ the points on $x \pm iy = 0$ equianharmonic to $(0, 1, 0)$ and $(0, 0, 1)$ with respect to the four given points. These results are expressed in the following theorem:

(3) The fifteen other possible points of inflexion of a cubic with two fixed imaginary pairs are the three diagonal points of the fixed quadrangle, and the two points on every one of the six sides of the quadrangle equianharmonic to the diagonal point with respect to the two fixed points on that side.

The vertices of the three determined quadrangles may also be obtained analytically as the intersections of three conics. Three pairs of lines

$$y \pm iz = 0, \qquad z \pm x = 0, \qquad x \pm y = 0$$

are uniquely determined as being harmonic both with respect to the sides of the given quadrangle $(i, \pm 1, \pm 1)$ and with respect to the sides of the diagonal triangle. Three conics having respectively these pairs of lines as tangents, namely,

$$2x^2 - y^2 - z^2 = 0, \qquad x^2 - 2y^2 - z^2 = 0, \qquad x^2 - y^2 - 2z^2 = 0$$

pass, taken in the same order, through the pairs of quadrangles

$$\begin{array}{ll} (\pm 1, \sqrt{3}, \pm i), & (\pm 1, \pm i, \sqrt{3}); \\ (\pm 1, \pm i, \sqrt{3}), & (\sqrt{3}, \pm 1, \pm 1); \\ (\sqrt{3}, \pm 1, \pm 1), & (\pm 1, \sqrt{3}, \pm i). \end{array}$$

Further any one of the three conics passes through the four intersections

of the pairs of tangents given for the other two. Thus any one of these conics is uniquely determined by four points and two tangents, the determining elements being fixed by means of harmonic properties with respect to the given quadrangle. In turn the three conics uniquely determine the three quadrangles

$$(\sqrt{3}, \pm 1, \pm 1), \qquad (\pm 1, \sqrt{3}, \pm i), \qquad (\pm 1, \pm i, \sqrt{3}).$$

Accordingly the fifteen other possible points of inflexion of a cubic with two fixed imaginary pairs are the three diagonal points of the fixed quadrangle, and the twelve intersections of three conics uniquely determined analytically by the quadrangle.

(iii) *Actual Construction of the Points.*

It has been noted that when two pairs of imaginary points are given as the intersections of two real lines with a conic, there follows a quadratic construction for the triangle self-polar for the pencil of conics through the four points.* It will now be shown that with the help of the triangle, the fifteen other possible points of inflexion of a cubic inflected at the two pairs of imaginary points may be determined by a series of constructions of which one only is quadratic, the rest linear.

As A is the intersection of the two given lines, the given conic meets BC certainly and either AB or CA in real points. Let it meet CA and call the points D, D'. Denote the intersections of BC with the given lines by E, E'.

The given points being $(i, \pm 1, \pm 1)$, the object is to construct the lines

$$x + \sqrt{3}y = 0, \qquad x \pm \frac{1}{\sqrt{3}} y = 0, \qquad z \pm \sqrt{3}x = 0, \qquad z \pm \frac{1}{\sqrt{3}} x = 0.$$

The conic of the pencil through the four given points that meets $y = 0$ on the lines $z \pm \sqrt{3}x = 0$ is $3x^2 + by^2 - z^2 = 0$, with the condition that $-3 + b - 1 = 0$, that is, the conic

$$3x^2 + 4y^2 - z^2 = 0.$$

This conic meets $x = 0$ where $4y - z = 0$, hence the first step requires the construction of the points of intersection of the lines $2y - z = 0$, $2y + z = 0$ with BC. Since

$$\begin{array}{llll} 2y - z = 0, & z = 0; & y - z = 0, & y = 0, \\ 2y + z = 0, & z = 0; & y + z = 0, & y = 0 \end{array}$$

are two sets of harmonic lines, these points P_1, P_2 may be constructed

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* See page 262.

linearly. Draw P_1D intersecting the given conic in a second point D'' and the given lines in F, F'. As the conics of a pencil cut any line in involution, P_3 the conjugate to P_1 in the involution $(D''D, FF')$ is another point on the conic

$$3x^2 + 4y^2 - z^2 = 0.$$

This conic is a member of a second pencil through two points P_1 (AP_1 being a tangent at a given point), the point P_2,* and the point P_3. The pairs of lines

$$AP_1, \quad P_2P_3; \qquad P_1P_2, \quad P_1P_3$$

are two other conics of the second pencil: hence this pencil cuts out on CA the involution (AV, CD), where V is the intersection of P_2P_3 with CA. The involution cut out on CA by the first pencil (the pencil through the four given points) is defined by its two double points, and may be expressed as (A^2, C^2). It follows that Q, Q', the common points of the two involutions

$$(AV, CD), \qquad (A^2, C^2),$$

found by means of the given conic (the one quadratic construction), are the intersections of $3x^2 + 4y^2 - z^2 = 0$ with $y = 0$. Hence BQ, BQ' are the desired lines

$$z \pm \sqrt{3}x = 0.$$

Since

$$z - \frac{1}{\sqrt{3}}x = 0, \qquad z + \sqrt{3}x = 0; \qquad z = 0, \qquad z - \sqrt{3}x = 0,$$
$$z + \frac{1}{\sqrt{3}}x = 0, \qquad z - \sqrt{3}x = 0; \qquad z = 0, \qquad z + \sqrt{3}x = 0$$

are two sets of harmonic lines, two other of the desired lines,

$$z \pm \frac{1}{\sqrt{3}}x = 0,$$

may be constructed linearly: call them BK, BK'. The lines that join the intersections of $z \pm \sqrt{3}x = 0$, $z \pm \frac{1}{\sqrt{3}}x = 0$ with $y \pm z = 0$ to C are

$$x \pm \sqrt{3}y = 0, \qquad x \pm \frac{1}{\sqrt{3}}y = 0.$$

Hence the complete construction (Fig. 2) may be stated as follows:

Construct P_1, P_2 the harmonic conjugates of B with respect to E, C and E', C. Draw P_1D intersecting the given conic in a second

* The conic $3x^2 + 4y^2 - z^2 = 0$ also has AP_2 for a tangent, but the use of two points P_1 and two points P_2 would give an illusory construction.

point D'' and the line-pair in F, F'. Construct P_3 the conjugate to P_1 in the involution $(D''D, FF')$. Draw P_2P_3 intersecting CA in V. Determine Q, Q' the common points * for the two involutions $(AV,$

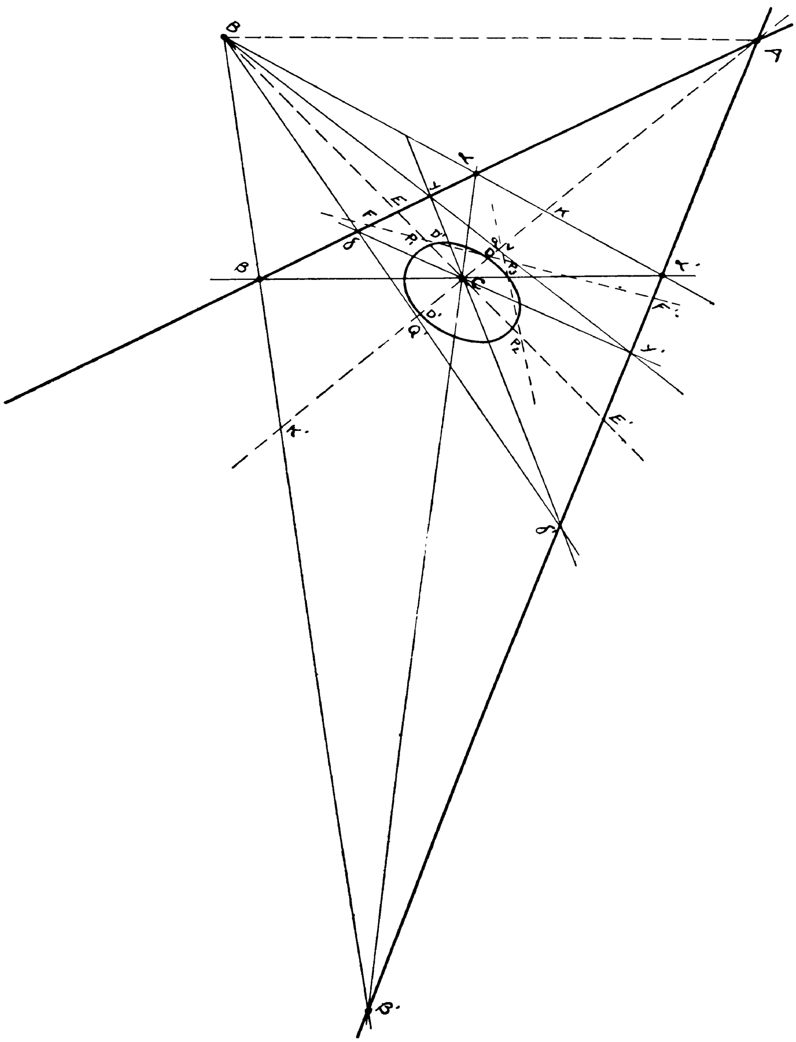

FIG. 2.

$CD)$, (A^2, C^2). Construct K, K' the harmonic conjugates of Q' with respect to A, Q and of Q with respect to A, Q'. Draw BK, BK', BQ, BQ' intersecting the given line-pair in α, α'; β, β'; γ, γ'; δ, δ'. Draw $\alpha\beta'$, $\alpha'\beta$, $\gamma\delta'$, $\gamma'\delta$.

The seven real points of inflexion are

$$A,\ B,\ C,\ \alpha,\ \alpha',\ \beta,\ \beta'.$$

* That Q, Q' are real is shown by the analytical discussion.

The eight imaginary points lie by pairs on the lines

$$\gamma\gamma',\ \delta\delta',\ \gamma\delta',\ \gamma'\delta,$$

where they are met by the two pairs of imaginary sides of the given quadrangle.

(iv) *Symmetrical Constructions.*

When the two pairs of imaginary inflexions are given as the intersections of two equal hyperbolas with the same pair of axes, the construction (Fig. 3) of the fifteen points is unique and furnishes an illustration of the three conics which intersect in the vertices of the determined quadrangles.

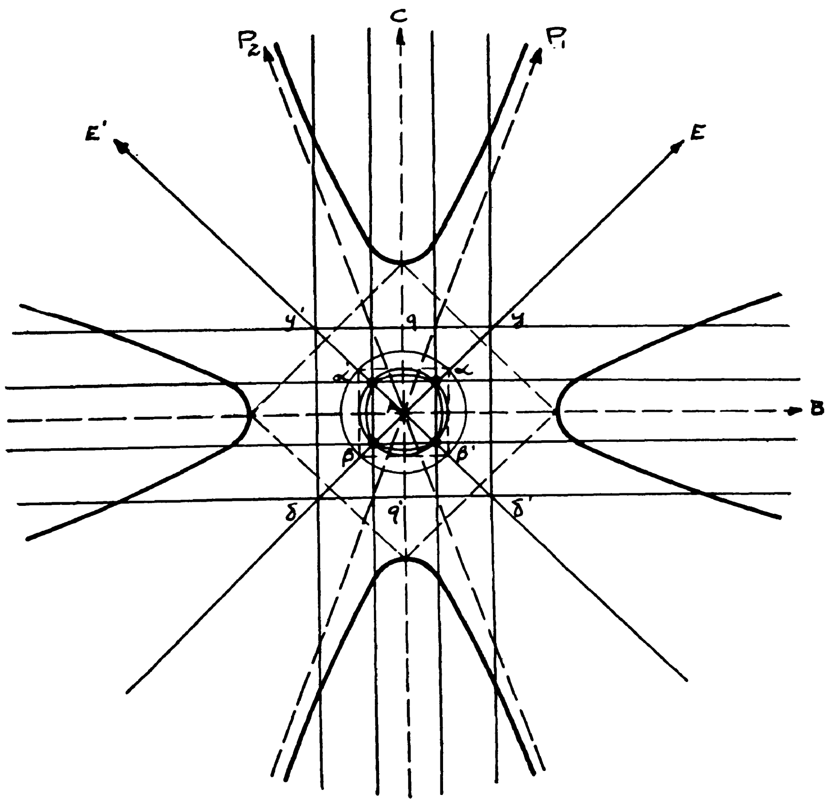

FIG. 3.

The hypothesis gives the axes and line at infinity, that is, the self-polar triangle; and (keeping the lettering of the preceding construction) the point A is the common center of the two hyperbolas. Draw the four other lines joining the vertices of the hyperbolas; then AE, AE', the real line-pair through the four given points,* bisect these lines. The lines AP_1, AP_2

* The equality of the hyperbolas accounts for the construction of these lines, in general not possible by quadratic construction when the two pairs of imaginary points are given by two conics.—See note, page 5.

are harmonic to AB with respect to AE, CA and AE', CA; and the points Q, Q' are determined as the vertices of the hyperbola through the four given points having AP_1, AP_2 as asymptotes.* Then the lines through Q, Q' parallel to AE together with the lines joining their intersections with the real line-pair are $\gamma\gamma'$, $\delta\delta'$, $\gamma\delta'$, $\gamma'\delta$; and by means of the harmonic relations between these lines and the axes AB, CA the lines $\alpha\alpha'$, $\beta\beta'$, $\alpha\beta'$, $\alpha'\beta$ may be constructed.

The above construction determines the vertices of the three quadrangles as the intersections of lines. To show them as the intersections of conics, let the two given hyperbolas be members of the pencil $ax^2 + by^2 + cz^2 = 0$, where $-a + b + c = 0$, when (1) $x = 0$ is the line at infinity and (2) $y = 0$, $z = 0$ and $y + z = 0$, $y - z = 0$ are two pairs of perpendicular lines. Then the three conics intersecting in the vertices of the determined quadrangles are two equal, symmetrically placed ellipses,

$$2y^2 + z^2 = x^2, \qquad y^2 + 2z^2 = x^2,$$

and the circle

$$y^2 + z^2 = 2x^2,$$

all three concentric with the hyperbolas.

Accordingly construct the two equal, symmetrically placed ellipses through α, α', β, β'; and pass a circle through the four finite intersections of the tangents to the ellipses at their vertices. Then the fifteen other possible points of inflexion of a cubic inflected at the four imaginary intersections of the two hyperbolas are the common center, the two points at infinity on the axes, the four real intersections of the two ellipses, and the eight imaginary intersections of the circle with the ellipses.

Another symmetrical construction is obtained by projecting one pair of the given points into the circular points. Then, the pencil of conics through the two pairs of imaginary points is a system of coaxial circles, the real line-pair consists of the radical axis and the line at infinity, and two vertices of the self-polar triangle are the limiting points of the system while the third vertex is at infinity on the radical axis. A pair of circles, each having one limiting point as a center and passing through the other limiting point, intersect on the radical axis in two vertices of the quadrangle of real points. A second pair of circles, having these vertices as centers and passing through the limiting points, determine four other points (z) on the first pair. The lines joining these four points to the limiting points pass through the remaining possible points of inflexion of cubics inflected at the

* Draw a line parallel to AP_1 intersecting the real line-pair and the hyperbola with AB as transverse axis. The center of the involution determined by the two pairs of points of intersection is a point on the hyperbola having AP_1 and AP_2 as asymptotes; and it is known that a hyperbola can be constructed when the asymptotes and one point on the curve are given.

four given imaginary points. This gives a unique construction when the two pairs of imaginary points are taken as the intersections of two circles. See (Fig. 4), where to complete the symmetry the two given circles are drawn equal.

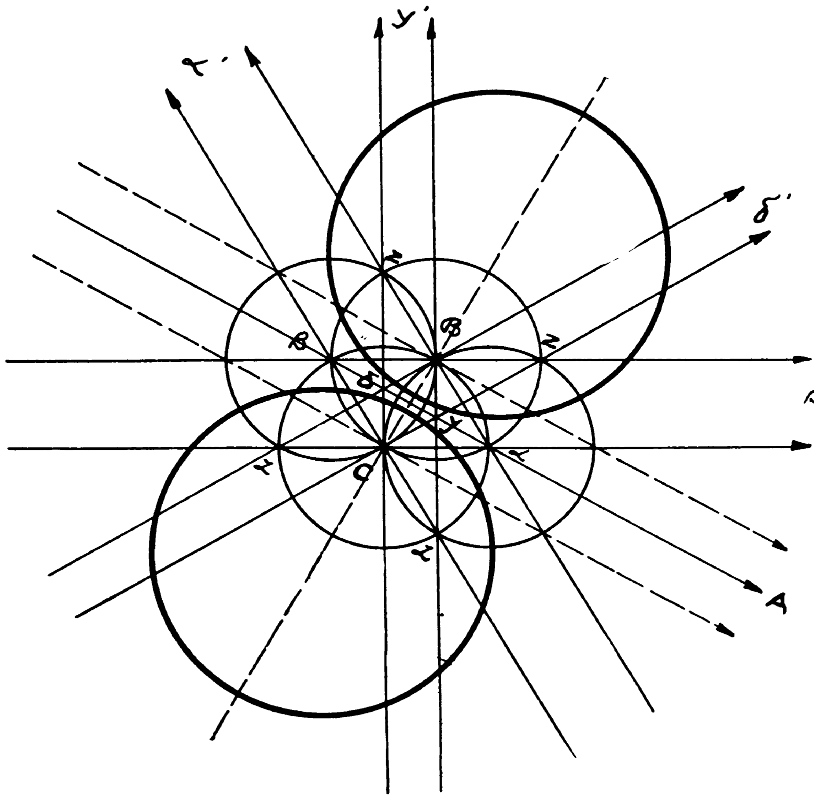

FIG. 4.

For the proof of the construction project ($\pm i$, 1, 1) into the circular points and change from homogeneous to Cartesian coördinates. The equation of the system of coaxial circles is then

$$x^2 + y^2 - 2\lambda y + \lambda = 0,$$

with $2y - 1 = 0$ as the radical axis and (0, 0), (0, 1) as the limiting points. The two circles each having one limiting point as center and passing through the other are

$$x^2 + y^2 = 1, \qquad x^2 + (y - 1)^2 = 1;$$

and these circles intersect on the radical axis in $(\pm \frac{1}{2}\sqrt{3}, \frac{1}{2})$, or $(\pm \sqrt{3}, 1, -1)$ in the homogeneous coördinates given by $z = y - 1$. The second pair of circles having these two points as centers and passing through the limiting points, namely,

$$(x - \tfrac{1}{2}\sqrt{3})^2 + (y - \tfrac{1}{2})^2 = 1, \qquad (x + \tfrac{1}{2}\sqrt{3})^2 + (y - \tfrac{1}{2})^2 = 1,$$

determine on the first pair the points

$$(\tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}), \qquad (-\tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}), \qquad (\tfrac{1}{2}\sqrt{3}, \tfrac{3}{2}), \qquad (-\tfrac{1}{2}\sqrt{3}, \tfrac{3}{2}),$$

or

$$(\sqrt{3}, -1, -3), \qquad (-\sqrt{3}, -1, -3), \qquad (\sqrt{3}, 3, 1), \qquad (-\sqrt{3}, 3, 1);$$

and the lines joining these four points to the limiting points are

$$x \pm \sqrt{3}y = 0, \qquad x \pm \frac{1}{\sqrt{3}}y = 0, \qquad z \pm \sqrt{3}x = 0, \qquad z \pm \frac{1}{\sqrt{3}}x = 0.$$

§ 2. GIVEN QUADRANGLE—TWO REAL AND A PAIR OF IMAGINARY POINTS.

Let four points, two real and one imaginary pair, to be taken as points of inflexion for a cubic be determined geometrically as the intersections of two real lines with a conic (Fig. 5). It is then known that the common

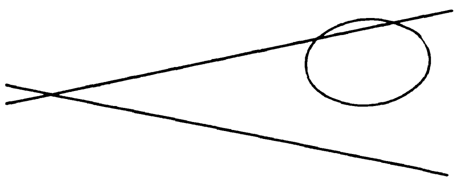

FIG. 5.

self-polar triangle for the pencil of conics through the four points has one real vertex, the intersection of the two given lines, and one real side, the polar of the real vertex with respect to the given conic; while the two remaining vertices and sides of the triangle are imaginary.

The study of the cubics with two real and a pair of imaginary inflexions fixed is correlated with the preceding study of the cubics with two fixed pairs of imaginary inflexions, by choosing the four points as $(\sqrt{3}, 1, \pm 1)$, $(1, \sqrt{3}, \pm i)$.*

(i) *Determination of the Five Points.*

The procedure followed in the case of the two pairs of imaginary points shows that the cubics with the four given inflexions have six common inflexional axes, namely, the three pairs of side of the given quadrangle,

$$x - \sqrt{3}y = 0, \qquad x - \frac{1}{\sqrt{3}}y = 0,$$

$$i\omega^2 x + \omega y \pm z = 0, \qquad -i\omega x + \omega^2 y \pm z = 0, \qquad (\omega^3 = 1).$$

Also as before a fifth point of inflexion is one of three: the real point A $(0, 0, 1)$ or either of the pair of imaginary points B $(i, \omega, 0)$, C $(-i, \omega^2, 0)$.

* See page 263.

Consider first a cubic with the real point A as a fifth point of inflexion. The cubic has two imaginary inflexional axes through this point; and the equation is consequently of the form

$$(i\omega^2x + \omega y + z)(i\omega^2x + \omega y - z)(x + \alpha y) \\ + \lambda(-i\omega x + \omega^2y + z)(-i\omega x + \omega^2y - z)(x + \beta y) = 0,$$

where α, β are a pair of complex numbers. Accordingly the remaining four inflexions are at

$$(\alpha, -1, i\omega\alpha + \omega^2), \qquad (\alpha, -1, -i\omega\alpha - \omega^2),$$
$$(\beta, -1, i\omega^2\beta - \omega), \qquad (\beta, -1, -i\omega^2\beta + \omega),$$

where, in order to satisfy the conditions of collinearity imposed on every group of inflexions of a non-singular cubic, that is, the conditions represented by the scheme

$$\begin{array}{ccc} (\sqrt{3}, 1, 1), & (\beta, -1, -i\omega^2\beta + \omega), & (\alpha, -1, i\omega\alpha + \omega^2), \\ (\beta, -1, i\omega^2\beta - \omega), & (\sqrt{3}, 1, -1), & (\alpha, -1, -i\omega\alpha - \omega^2), \\ (1, \sqrt{3}, i), & (1, \sqrt{3}, -i), & (0, 0, 1) \end{array}$$

either $\alpha = -i$, $\beta = i$, or $\alpha = \frac{1}{7}(-i - 4\sqrt{3})$, $\beta = \frac{1}{7}(i - 4\sqrt{3})$.

If $\alpha = -i$, $\beta = i$, the four points are $(i, \pm 1, \pm 1)$ in agreement with the results in the preceding case. If $\alpha = \frac{1}{7}(-i - 4\sqrt{3})$, $\beta = \frac{1}{7}(i - 4\sqrt{3})$, a second set of four points is obtained; but since the computations involved are complicated it is advantageous to apply a linear transformation by which the original four points become

$$(0, 1, -1), \qquad (-1, 0, 1), \qquad (1, -\omega, 0), \qquad (1, -\omega^2, 0).$$

Then the fifth point is $(1, -1, 0)$; the remaining four are either

$$(0, 1, -\omega), \qquad (0, 1, -\omega^2), \qquad (-\omega, 0, 1), \qquad (-\omega^2, 0, 1),$$

or

$$(-1, \omega^2 - 1, 1), \ (-1, \omega - 1, 1), \ (\omega^2 - 1, -1, 1), \ (\omega - 1, -1, 1);$$

and the corresponding pencils of cubics may be written as

$$x^3 + y^3 + z^3 + \lambda xyz = 0,$$
$$x^3 + y^3 + z^3 + 3z(x^2 + y^2) + 3z^2(x + y) + \lambda z(z + x)(y + z) = 0.$$

Similarly if either B or C is the fifth point of inflexion, there are two distinct pencils of cubics; but, since the four pencils thus determined are imaginary and consequently of interest in this discussion only to give symmetry to the results, their equations and the coördinates of their remaining points of inflexion are omitted.

The cubics of the three pairs of pencils have in all only nineteen points of inflexion, and these have the same relative positions as the nineteen for the three pairs of pencils determined by two pairs of imaginary inflexions. Hence, it follows that, with the proper interchanging of the words real and imaginary, the theorems stated on pages 264, 265, and 266 hold for the cubics with two real and an imaginary pair of fixed inflexions. The constructions, however, because of the great number of imaginary elements involved become almost entirely theoretical.

(ii) *Related Quartic Curves.*

The equations

$$P_1 \equiv x^3 + y^3 + z^3 + \lambda xyz = 0,$$
$$P_2 \equiv x^3 + y^3 + z^3 + 3z(x^2 + y^2) + 3z^2(x + y) + \lambda z(z + x)(x + y) = 0$$

represent two pencils of cubics having in common three real and a pair of imaginary inflexions. Every value of λ determines a definite curve in each pencil, and the elimination of λ between the two equations gives

$$z^2[(x^3 + y^3 + z^3)(x + y + z) - 3xy(x^2 + y^2) - 3xyz(x + y)] = 0$$

as the locus of the intersections of the two curves. Three of the fixed intersections, namely, one real and the given pair of imaginary inflexions, are on $z = 0$; hence this line is a part of the locus only because of the two curves of which it forms a part. The remaining two fixed inflexions and the four variable intersections lie on the quartic

$$(x^3 + y^3 + z^3)(x + y + z) - 3xy(x^2 + y^2) - 3xyz(x + y) = 0.$$

The remaining two fixed inflexions are $(0, 1, -1)$, $(-1, 0, 1)$, where the quartic has nodes with tangents

$$\pm ix + y + z = 0, \qquad x \pm iy + z = 0;$$

and these lines are the inflexional tangents common to the two cubics considered when $\lambda = \pm 3i$. Thus the binodal quartic is the locus of the four variable interesections of the two curves of P_1 and P_2 having the same parametric value.

The two sets of four points

$$(0,\ 1,\ -\omega), \quad (0,\ 1,\ -\omega^2), \quad (-\omega,\ 0,\ 1), \quad (-\omega^2,\ 0,\ 1);$$
$$(-1,\ \omega^2 - 1,\ 1), \ (-1,\ \omega - 1,\ 1), \ (\omega^2 - 1,\ -1,\ 1), \ (\omega - 1,\ -1,\ 1),$$

completing the inflexional groups of P_1 and P_2 are on the quartic; and these together with the two double points $(0, 1, -1)$, $(-1, 0, 1)$ are the complete intersection of two quartics

$$(\omega x + y + z)(\omega^2 x + y + z)(x + \omega y + z)(x + \omega^2 y + z) = 0,$$
$$xy(z + x)(x + y) = 0,$$

each composed of two pairs of lines, one pair through each of the double points. Accordingly

$$(\omega x + y + z)(\omega^2 x + y + z)(x + \omega y + z)(x + \omega^2 y + z) \\ + \mu xy(z + x)(x + y) = 0$$

is a pencil of binodal quartics through the eight points and two nodes. If the second pencil of cubics is written

$$P_2 \equiv x^3 + y^3 + z^3 + 3z(x^2 + y^2) + 3z^2(x + y) + (\lambda - n)z(z + x)(y + z) = 0,$$

where n has any real value, a different quartic of the pencil corresponds to each chosen value of n, that is, the manner of writing the equations of P_1 and P_2 can be so varied that every quartic of the pencil may be found as the locus of the four variable intersections of the two cubics with the same parametric value. The two reducible quartics corresponding to μ infinite or zero are obtained respectively when $n = \infty$ or $n = 3$. If $n = \infty$, P_2 breaks up into three lines. If $n = 3$, the two cubics have the same tangents at the two common imaginary inflexions $(1, -\omega, 0)$, $(1, -\omega^2, 0)$ for every value of λ; and all their intersections lie at the five given points except when the cubics have a common linear factor. In the study of non-singular cubics these two cases are excluded, and hence the following result can be stated:

There are two, and only two, pencils of cubics having in common three real and a pair of imaginary inflexions; and the locus of the four variable intersections of the two corresponding curves is a binodal quartic passing through the remaining inflexions of the two pencils, the two nodes being at the two real inflexions not collinear with the two common imaginary inflexions.

In further consideration of the geometry on the pairs of cubics and the resulting quartic curve it may be noted that three of the nine intersections of the two cubics lie on a line, hence the remaining six, the six on the quartic, lie on a conic. The equation

$$P_1 - P_2 \equiv 3z[x^2 + y^2 + z(x + y) + \frac{\lambda}{3}z(x + y + z) = 0$$

represents a pencil of cubics consisting of the line and a pencil of conics. Every value of λ determines a curve of P_1 and P_2 and a conic through their six intersections on the quartic. The pencil of conics passes through the two nodes $(0, 1, -1)$, $(-1, 0, 1)$, and the two points $(1, \pm i, 0)$ where the line $z = 0$ is intersected by the nodal tangents

$$\pm ix + y + z = 0, \qquad x \pm iy + z = 0.$$

Thus the four variable intersections of the two cubics are cut out on the

binodal quartic by a pencil of concis through the two nodes and the two intersections of the nodal tangents collinear with the two common imaginary inflexions.

When the points $(1, \pm i, 0)$ are projected into the circular points, the special case arises where the four variable intersections lie on a circle through two of the common real points of inflexion. It may also be noticed that, since the four variable intersections of the two cubics and the two nodes of the quartic lie on a conic, the four variable intersections subtend at the two nodal points pencils of lines with the same cross-ratio.

§ 3. GENERAL CONCLUSIONS.

For symmetry the cubics with a fixed quadrangle of real points of inflexion are considered, although every such cubic is imaginary. Choose the four points $(\sqrt{3}, \pm 1, \pm 1)$. Then the six fixed inflexional axes are

$$y \pm z = 0, \qquad z \pm \frac{1}{\sqrt{3}}x = 0, \qquad x \pm \sqrt{3}y = 0;$$

and a fifth point of inflexion is any one of the three

$$(1, 0, 0), \qquad (0, 1, 0), \qquad (0, 0, 1).$$

It follows that a cubic of the system with an inflexion at $(1, 0, 0)$ is a member of one of the pencils

$$(x^2 - 3y^2)(y + i\sqrt{3}z) + \lambda(z^2 - \tfrac{1}{3}x^2)\left(y + \frac{i}{\sqrt{3}}z\right) = 0,$$

$$(x^2 - 3y^2)(y - i\sqrt{3}z) + \lambda(z^2 - \tfrac{1}{3}x^2)\left(y - \frac{i}{\sqrt{3}}z\right) = 0;$$

and similar results hold with respect to $(0, 1, 0)$ and $(0, 0, 1)$. Furthermore the six pencils have nineteen points of inflexion associated as in the two preceding cases. Hence the theorems already stated (pp. 264, 265, 266) are applicable to this case also, that is, for cubics with any fixed quadrangle of inflexions* the following theorems hold:

† (1) If four points of inflexion of a plane cubic, no three collinear, are fixed, a fifth point of inflexion is fixed as one of three, and the complete group of nine is determined as one of six; or in other words,

(1′) A quadrangle of inflexions determines a system of cubics consisting of six syzygetic pencils, and every two of the six have a fifth point of inflexion in common.

* Provided, as stated on page 261, that if imaginary the points enter by conjugate pairs.

† A. Wiman, Nyt Tiddskrift for Matematic (1894); also W. Burnside, *Proc. London Math. Soc.* (1906–07).

(2) The remaining points of inflexion of the cubics with a common quadrangle of inflexions form a group of fifteen, consisting of the three diagonal points of the common quadrangle and the vertices of three other quadrangles with the same diagonal points, each of the three new quadrangles having one pair of sides in common with the original quadrangle.

(3) The fifteen other possible points of inflexion of a cubic with a fixed quadrangle of inflexions are the three diagonal points of the fixed quadrangle, and the two points on every one of the six sides of the quadrangle equianharmonic to the diagonal point with respect to the two fixed points on that side.